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Statistical Properties of the Odd Binomial States with Dynamical Applications

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In this paper we investigate the odd binomial state, which interpolates between odd number and odd coherent states. We discuss the statistical properties of the Glauber second-order correlation function and squeezing phenomena (normal squeezing and amplitude-squared squeezing). The phase properties in Pegg–Barnett formalism are considered, and the quasiprobability distribution functions are examined. The dynamics of the Jaynes–Cummings model and the resonance fluorescence are also discussed.

1. INTRODUCTION

There recently has been a considerable effort to generate new quantum states besides the number states and the coherent states. In fact most of these states interpolate between the number state and the coherent state, or between the number state and the thermal state. For example, the binomial state interpolates between the number state and the coherent state.(1) Also we find the negative binomial state tending for some limiting cases to the coherent state, or to the pure thermal states, depending upon the value of the parameters; for more details see refs. 2–5. Furthermore, the negative binomial state, which exhibits some nonclassical effects, tends to the logarithmic state if one removes the vacuum state $|0\rangle$.(6) As another example we mention the

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generalized geometric state, which interpolates between the number state and the (nonpure) chaotic state. This state has been considered extensively in refs. 7–9. Recently, the superposition of two coherent states has been investigated with the aim of understanding the rate of quantum interference between coherent states and consequently to generate states whose properties are different from an ordinary coherent state.(10) In fact quantum interference between coherent states leads to a reduction of quadrature fluctuations as well as a reduction of photon number fluctuations. The fundamental properties of quantum mechanical superposition states of light have attracted attention, and the possibility of generating such states in experiments has been proposed by many authors. For example, Yurke and Stoler(11) have shown that a coherent state propagating through an amplitude-dispersive medium, under certain specified conditions of parameters, can evolve into a superposition of two coherent states 180° out of phase.

Also, with the micromaser we can see interaction between the cavity field and a sequence of atoms, and can produce superposition states of two coherent states, namely even and odd coherent states.(12) These superposition states exhibit oscillations in their photon-number distributions and other nonclassical properties, such as squeezing and sub-Poissonian photon statistics. Therefore it would be worthwhile to consider more widely classes of states which interpolate between the even (odd) coherent state and the even (odd) number state, such as the even (odd) binomial state. In previous communications(13,14) we have considered the even binomial state, which interpolates between the even coherent and the even number states; therefore our aim in the present paper is to consider the odd binomial state, which represents a different type of intermediate state. This state interpolates between the odd number state and odd coherent state and is defined by

$$\begin{aligned} |\psi_0\rangle &= \frac{\lambda}{2}[|\eta, M\rangle - |-\eta, M\rangle] \\ &= \lambda \sum_{n=0}^{[(M-1)/2]} \binom{M}{2n+1}^{1/2} \eta^{2n+1}(1-|\eta|^2)^{(M-2n-1)/2}|2n+1\rangle \end{aligned} \tag{1.1}$$

where λ is the normalization constant given by

$$|\lambda|^2 = 2[1-(1-2|\eta|^2)^M]^{-1} \tag{1.2}$$

In equation (1.1) $|\eta, M\rangle$ represents the binomial state defined by

$$|\eta, M\rangle = \sum_{n=0}^{M} B_n^M \eta^n (1-|\eta|^2)^{(M-n)/2}|n\rangle, \qquad B_n^M = \sqrt{\binom{M}{n}}$$

Since the linear combination of the odd binomial state does not contain the vacuum state, the range of the parameter η will be between 0 and 1 such that $0 < |\eta| \leq 1$. Now if one takes the limit $M \to \infty$ and $\eta \to 0$ such that

$$\lim_{(M,\eta;M\eta)\to(\infty,0,\alpha)} |\psi_0\rangle = (\operatorname{csch}|\alpha|^2)^{1/2} \sum_{n=0}^{\infty} \frac{\alpha^{(2n+1)}}{\sqrt{(2n+1)!}} |2n+1\rangle \quad (1.3)$$

then we find equation (1.3) represents the usual definition of the odd coherent state.[15] In ref. 16 the state given by equations (1.1) is generated through the successive application of a classical laser field and a quantized cavity field.

The organization of the paper is as follows: In section 2 we consider and discuss the sub-Poissonian behavior and squeezing phenomena. Section 3 is devoted to phase properties in Pegg–Barnett formalism followed; the quasiprobability distribution functions are treated in Section 4, while in Section 5 we examine the Jaynes–Cummings model (i.e., the model of a two-level atom interacting with a single-mode field in a perfect cavity) against the odd binomial state; we also extend our discussion to include the resonance fluorescence for a single atom and many cooperative atoms. Finally we give our conclusions in Section 6.

2. SUB-POISSONIAN BEHAVIOR AND SQUEEZING PHENOMENA FOR ODD BINOMIAL STATES

In this section we consider in detail the correlation function as well as squeezing phenomena. We have first to calculate the expectation values of powers of the annihilation (creation) operator $a(a^\dagger)$ with respect to the odd binomial state. Due to the nature of the photon number distribution, one would expect the values of various odd powers of $a(a^\dagger)$ and any power of these operators higher than M to vanish. Therefore we have the following expressions:

$$\langle a^{2s}\rangle = \frac{|\lambda|^2\eta^{2s}}{(1-|\eta|^2)^s} \sum_{n=0}^{[(M-1)/2]-s} |B_{2n+1}^M|^2 \sqrt{\frac{(M-2n-1)!}{(M-2n-2s-1)!}} \quad (2.1)$$

and the expectation value of the photon number is

$$\langle a^\dagger a\rangle = |\lambda|^2|\eta|^2(M/2)[1 + (1-2|\eta|^2)^{(M-1)}] \quad (2.2)$$

where s is a positive integer and a and $a^\dagger$ satisfy the commutation relation $[a, a^\dagger] = 1$. The difference between the expectation value of the second moments of the photon number and the expectation value of the photon number itself is given by

$$\langle a^{\dagger 2} a^2 \rangle = \frac{1}{2} |\lambda|^2 |\eta|^4 M(M - 1) [1 - (1 - 2|\eta|^2)^{M-2}] \tag{2.3}$$

2.1. Normalized Second-Order Correlation Function

In this subsection we shall employ the Glauber second-order correlation function to discuss some statistical properties such as a sub-Poissonian distribution,[17,18] which is characteristic of the odd binomial state. The Glauber second-order zero-time correlation function is defined as

$$g^{(2)}(0) = \frac{\langle a^{\dagger 2} a^2 \rangle}{\langle a^{\dagger} a \rangle^2} \tag{2.4}$$

From equations (2.2)–(2.4) we have

$$g^{(2)}(0) = 1 - 1/M + 4(1 - 1/M) \frac{(1 - 2|\eta|^2)^{M-2}(1 - |\eta|^2)^2}{[1 + (1 - 2|\eta|^2)^{(M-1)}]^2} \tag{2.5}$$

Equation (2.5) shows that the value of the correlation function $g^{(2)}(0)$ is always less than one insofar as both η and M are finite. However, if we increase the value of M and decrease the value of η at the same time, such that $\eta \to 0$ as $M \to \infty$, then we find the correlation function tends to $\tanh^2|\alpha|^2$, so that a sub-Poissonian effect does exist. This emphasizes that the odd binomial state has sub-Poisonnian behavior. The gradual behavior from the odd number state to the odd coherent state can be seen in Fig. 1, where the function $g^{(2)}(0)$ is plotted against the parameter η (η is taken to be real) for different values of the parameter M. We can easily see that for large values of M and fixed value of η, the function $g^{(2)}(0)$ approaches unity more rapidly as $\eta \to 1$ and persists, where the system in this case shows coherence behavior.

2.2. Normal Squeezing

The squeezing phenomenon represents one of the interesting phenomena in the field of quantum optics, and is a direct quantum effect of Heisenberg's uncertainty principle. It reflects the reduced quantum fluctuations in one of the field quadratures at the expense of the other corresponding stretched quadrature.

The investigation of normal squeezing is based on defining two field quadrature operators

$$\hat{X}_1 = \frac{1}{2}(a + a^{\dagger}), \qquad \hat{X} = \frac{1}{2i}(a - a^{\dagger}) \tag{2.6}$$

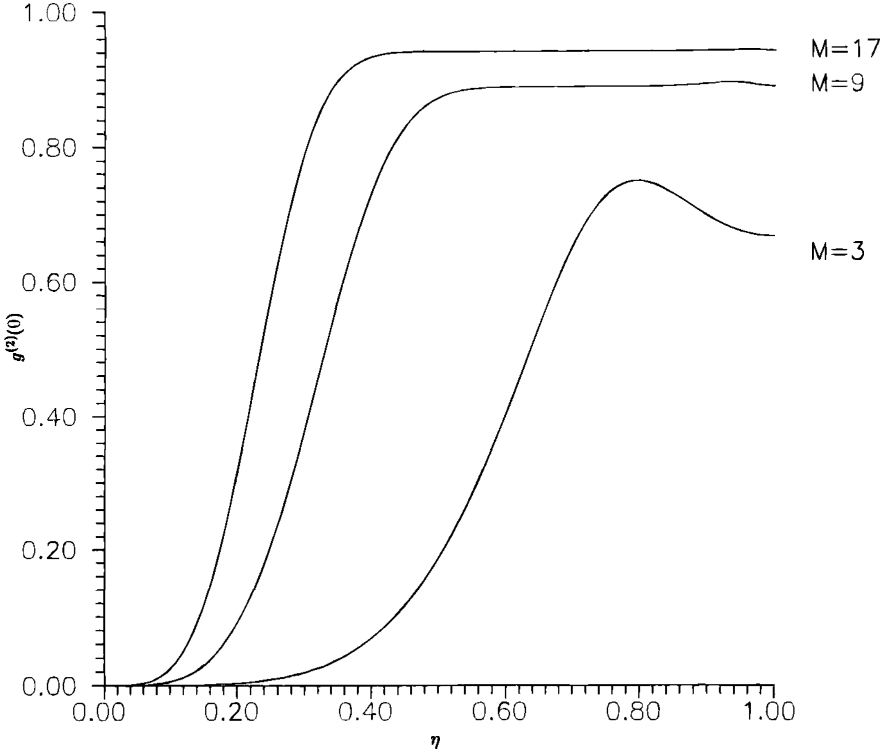


Fig. 1. $g^{(2)}(0)$ from equation (2.5).

These quadrature operators satisfy the commutation relation $[\hat{X}_1, \hat{X}_2] = i/2$. In order to see if the system acquires squeezing or not, we have to calculate the quadrature variance $(\Delta X_i)^2$, which is defined by

$$(\Delta \hat{X}_i)^2 = \langle \hat{X}_i^2 \rangle - \langle \hat{X}_i \rangle^2 \tag{2.7}$$

Then normal squeezing holds if

$$S_i = 4(\Delta \hat{X}_i)^2 - 1 < 0, \qquad i = 1 \text{ or } 2 \tag{2.8}$$

From equations (2.1) and (2.2) we find that (2.8) yields

$$S_1 = \frac{M}{2}|\lambda|^2|\eta|^2[1 + (1 - 2|\eta|^2)^{M-1}] + \frac{2M|\lambda|^2|\eta|^2 \cos 2\phi}{(1 - |\eta|^2)} \times \sum_{n=0}^{[(M-3)/2]} \sqrt{\left(1 - \frac{2n+2}{M}\right)\left(1 - \frac{2n+1}{M}\right)}\, |B_{2n+1}^M|^2 \tag{2.9a}$$

$$S_2 = \frac{M}{2}|\lambda|^2|\eta|^2[1 + (1 - 2|\eta|^2)^{M-1}] - \frac{2M|\lambda|^2|\eta|^2 \cos 2\phi}{(1 - |\eta|^2)}$$
$$\times \sum_{n=0}^{[(M-3)/2]} \sqrt{\left(1 - \frac{2n+2}{M}\right)\left(1 - \frac{2n+1}{M}\right)}\,|B_{2n+1}^M|^2 \qquad (2.9b)$$

where $\eta = |\eta|e^{i\phi}$, and ϕ is a phase parameter. From our numerical investigation we may conclude that the odd binomial state does not show squeezing whatever the values of $|\eta|$ and M are. The odd coherent state can be obtained from equations (2.9a) and (2.9b) as

$$(\Delta\hat{X}_1)^2 = \frac{1}{4} + \frac{|\alpha|^2}{2}(\cos 2\phi + \coth|\alpha|^2) \qquad (2.10a)$$

$$(\Delta\hat{X}_2)^2 = \frac{1}{4} + \frac{|\alpha|^2}{2}(\coth|\alpha|^2 - \cos 2\phi) \qquad (2.10b)$$

which are identical with those obtained in refs. 17 and 18.

2.3. Amplitude-Squared Squeezing

Now we use the concept of amplitude-squared squeezing introduced by Hillery.[20] This type of squeezing arises in a natural way in second-harmonic generation and in a number of nonlinear optical processes.

To study amplitude-squared squeezing for the odd binomial state we introduce the field quadrature operators as follows:

$$\hat{Y}_0 = \frac{1}{4}(aa^\dagger + a^\dagger a) \qquad (2.11a)$$

$$\hat{Y}_1 = \frac{1}{4}(a^2 + a^{\dagger 2}) \qquad (2.11b)$$

$$\hat{Y}_2 = \frac{1}{4i}(a^2 - a^{\dagger 2}) \qquad (2.11c)$$

Operators $\hat{Y}_1$ and $\hat{Y}_2$ satisfy the commutation relation

$$[\hat{Y}_1, \hat{Y}_2] = i\hat{Y}_0 \qquad (2.12)$$

so that the uncertainty principle applied to $\hat{Y}_1$ and $\hat{Y}_2$ is

$$(\Delta\hat{Y}_1)^2(\Delta\hat{Y}_2)^2) \geq \frac{1}{4}\langle\hat{Y}_0\rangle^2 \qquad (2.13)$$

Amplitude-squared squeezing holds if

$$Q_1 = (\Delta \hat{Y}_1)^2 - \frac{1}{2}\left|\langle \hat{Y}_0 \rangle\right| < 0 \tag{2.14a}$$

or

$$Q_2 = (\Delta \hat{Y}_2)^2 - \frac{1}{2}\left|\langle \hat{Y}_0 \rangle\right| < 0 \tag{2.14b}$$

From equations (2.1)–(2.3) together with equation (2.14a) we plot Q_1 against the parameter η (Fig. 2). In this figure we see that as M increases, the squeezing gets more pronounced, and the maximum point of squeezing

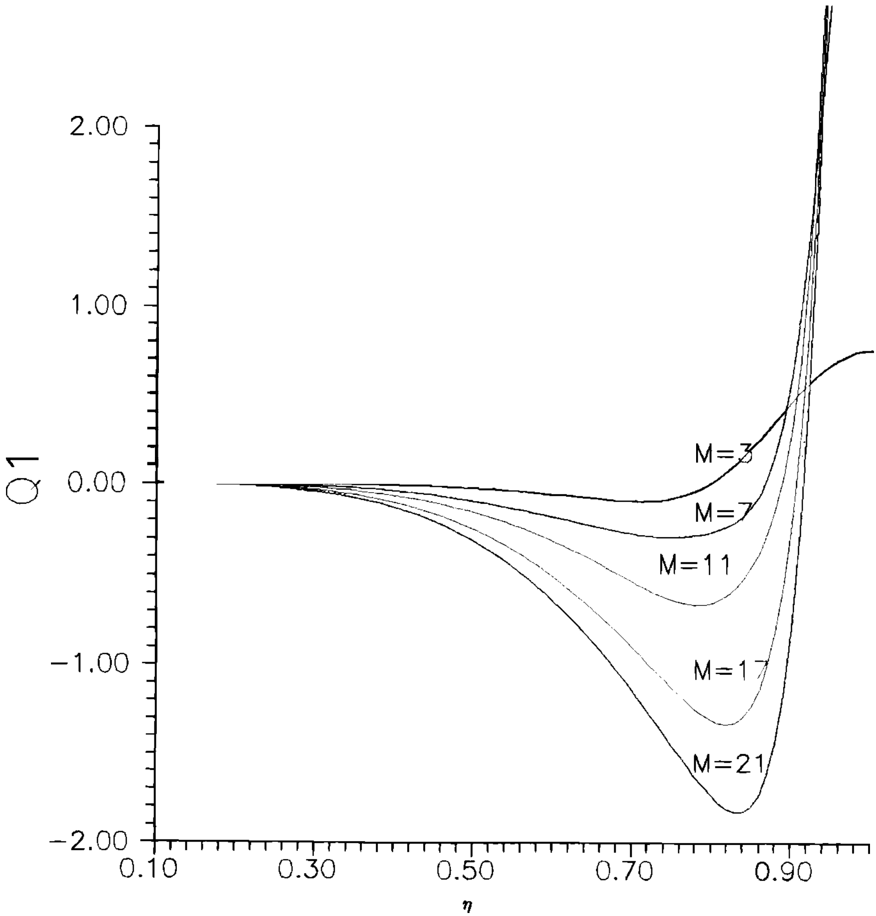


Fig. 2. Q_1 from equation (2.14a).

moves toward higher values of η. In the case of the odd coherent state, we have $(\Delta\hat{Y}_1)^2 = (\Delta\hat{Y}_2)^2$ such that

$$(\Delta\hat{Y}_1)^2 = (\Delta\hat{Y}_2)^2 = \frac{1}{8} + \frac{|\alpha|^2}{4}\coth|\alpha|^2 \tag{2.15}$$

Equation (2.15) ensures that the odd coherent state is a minimum uncertainty relation for the quadrature variances $\Delta\hat{Y}_1$ and $\Delta\hat{Y}_2$.

3. PHASE PROPERTIES

Since the concept of Hermitian phase operators plays an important role in quantum optics, we shall study the phase properties of the odd binomial state using the Pegg–Barnett formalism based on the phase states $|\Theta\rangle$ defined by

$$|\Theta_m\rangle = \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}\exp(in\Theta_m)|n\rangle \tag{3.1}$$

Equation (3.1) gives a complete set of $s+1$ orthonormal phase states provided

$$\Theta_m = \Theta_0 + \frac{2\pi m}{s+1}; \qquad m = 0, 1, \ldots, s \tag{3.2}$$

where the value of Θ_0 is arbitrary, and is taken here to be zero. In fact the phase states $|\Theta_m\rangle$ are eigenstates of the Hermitian phase operator $\hat{\phi}_\Theta$, which is defined by

$$\hat{\phi}_\Theta = \sum_{m=0}^{s}\Theta_m|\Theta_m\rangle\langle\Theta_m| \tag{3.3}$$

The state of the form

$$|b\rangle = \sum_{n=0}^{s} b_n e^{in\chi}|n\rangle \tag{3.4}$$

is known as a partial phase state,[20] where b_n are real and positive, and χ is a phase. From equations (3.1) and (3.4), we can calculate the expectation value and the variance for the phase operator $\hat{\phi}_\theta$ with respect to the partial phase state; we have

$$\langle\hat{\phi}_\theta\rangle = \chi \tag{3.5}$$

$$\langle\triangle\hat{\phi}_\Theta^2\rangle = \frac{\pi^2}{3} + 4\sum_{n>m}\frac{(-1)^{(n-m)}\, b_n b_m}{(n-m)^2} \tag{3.6}$$

The phase probability distribution for the partial phase state is given by

$$P(\Theta) = \left|\langle\Theta_m|b\rangle\right|^2 \tag{3.7}$$

Since the density of phase states is $(s + 1)/2\pi$, then in the continuum limit $s \to \infty$, equation (3.7) becomes

$$P(\Theta) = \frac{1}{2\pi}(1 + 2\sum_{n>m} b_m b_n \cos[(n - m)\Theta]) \tag{3.8}$$

In the case of the odd binomial state we have $\chi = \phi$ and

$$b_n = |\eta|^{2n+1}\sqrt{\binom{M}{2n+1}}(1 - |\eta|^2)^{[(M-2n-1)/2]} \tag{3.9}$$

In Fig. 3 we plot $P(\Theta)$ given by (3.8) against the parameter η, for $M = 17$. In this figure we can see that there is only one stretching peak along the η-axis at $\Theta = 0$ and two wings when Θ approaches $\pm\pi$. This situation is

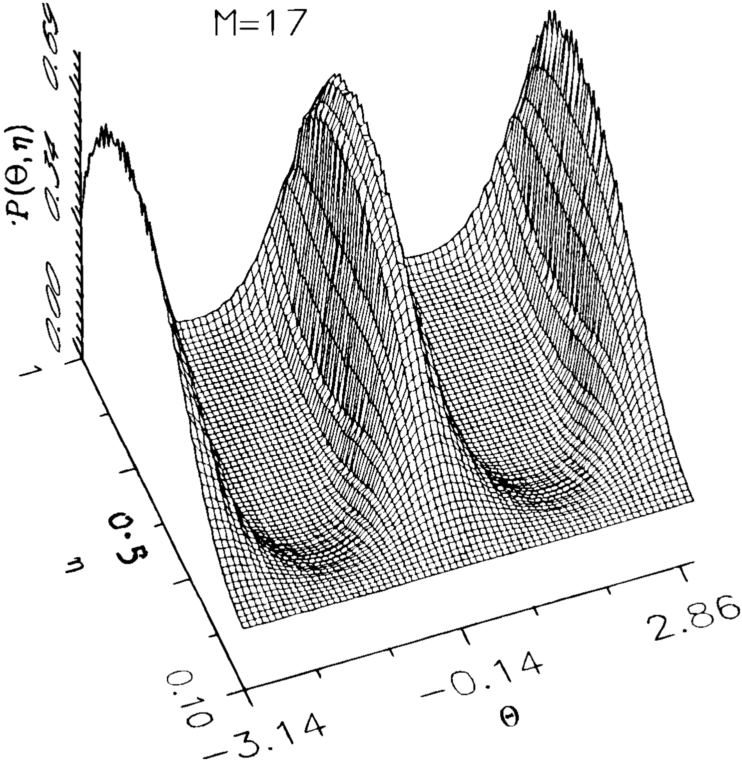


Fig. 3. $P(\Theta)$ from equation (3.8).

different compared with the case of the phased orthogonal even binomial state, where four peaks are involved owing to the presence of the number states $|4n\rangle$.[23]

4. QUASIPROBABILITY DISTRIBUTION FUNCTIONS

The quasiprobability distribution functions are important tools to discuss the statistical description of a microscopic system, and also to provide insight into the nonclassical features of the radiation field. In the present section we turn our attention to examine the quasiprobability distribution functions against the odd binomial state.

There are three types of these functions: P-function (Glauber–Sudershan), W-function (Wigner), and Q-function (Husimi). However, the integration of these functions runs over phase-space variables, and therefore it is not an easy task to calculate them. In order to find the quasiprobability distribution functions we have first to calculate the characteristic function with respect to the odd binomial state. The characteristic function $C_P(\beta)$ is defined by

$$C_p(\beta) = \mathrm{Tr}\,[\hat{\rho} e^{\beta a^\dagger} e^{-\beta^* a}] \tag{4.1}$$

where $\hat{\rho}$ is the density matrix for the odd binomial state given by

$$\hat{\rho} = \sum_{n,m}^{[(M-1)/2]} B_{2n+1}^M (B_{2m+1}^M)^* |2n+1\rangle\langle 2m+1| \tag{4.2}$$

Now if we use the relation[24]

$$\begin{aligned}\langle n|e^{\beta a^\dagger}\, e^{-\beta^* a}|m\rangle &= \sqrt{\frac{n!}{m!}}\,(-\beta^*)^{m-n} L_n^{(m-n)}(|\beta|^2), \qquad m \geq n \\ &= \sqrt{\frac{m!}{n!}}\,\beta^{n-m} L_m^{(n-m)}(|\beta|^2), \qquad m \leq n\end{aligned} \tag{4.3}$$

where $L_n^{\gamma}(|\beta|^2)$ and $L_n(|\beta|^2)$ are the associated Laguerre and Laguerre polynomials respectively, then the charateristic function given by equation (4.1) takes the form

$$\begin{aligned}C_P(\beta) = {} & |\lambda|^2 \sum_{n=0}^{[[M-1]/2]} |B_{2n+1}^M|^2 L_{2n+1}(|\beta|^2) \\ & + 2|\lambda|^2 \sum_{\substack{m,n=0\\ m>n}}^{[(M-1)/2]} \sqrt{\frac{(2m+1)!}{(2n+1)!}}\, |B_{2n+1}^M| \cdot |B_{2m+1}^M| \\ & \times |\beta|^{2(n-m)} L_{2n+1}^{2(m-n)}(|\beta|^2)\cos[2(n-m)(\phi+\zeta)]\end{aligned} \tag{4.4}$$

where $\beta = |\beta|e^{i\zeta}$. The above equation consists of two parts; the first part represents the diagonal term, while the second part represents the off-diagonal terms of the density matrix $\hat{\rho}$. Having obtained the characteristic function $C_P(\beta)$, we are in a position to calculate the quasiprobability distribution functions. These functions can be obtained if one calculates the following integral:

$$F(s, \alpha) = \pi^{-2} \int_{-\infty}^{\infty} d^2\beta \; C_P(\beta) \exp[\alpha\beta^* - \beta\alpha^* - (s/2)|\beta|^2] \qquad (4.5)$$

For $s = 1$, we obtain the Wigner function, while the Q-function can be found when we take $s = 2$; when $s = 0$ equation (4.5) gives us the P-function. By inserting equation (4.4) into equation (4.5) and performing the integral over the whole complex β-plane, we have the following expressions:

For $s = 1$ (the Wigner function)

$$W(\alpha) = -2\frac{|\lambda|^2}{\pi} e^{-2|\alpha|^2} \left[\sum_{n=0}^{[(M-1)/2]} |B_{2n+1}^M|^2 L_{2n+1}(4|\alpha|^2) + 2 \sum_{\substack{m,n=0 \\ m>n}}^{[(M-1)/2]} \sqrt{\frac{(2m+1)!}{(2n+1)!}} |B_{2n+1}^M| \cdot |B_{2m+1}^M| \times (2|\alpha|)^{2(n-m)} L_{2n+1}^{2(m-n)}(4|\alpha|^2) \cos[2(n-m)(\phi+\zeta)] \right] \qquad (4.5)$$

For $s = 2$ (the Q-function)

$$Q(\alpha) = \frac{|\lambda|^2}{\pi} e^{-|\alpha|^2} \left[\sum_{n=0}^{[(M-1)/2]} |B_{2n+1}^M|^2 \frac{|\alpha|^{4n+2}}{(2n+1)!} + 2 \sum_{\substack{m,n=0 \\ m>n}}^{[(M-1)/2]} \frac{|\alpha|^{2n+2m+2}}{\sqrt{(2m+1)!(2n+1)!}} \times |B_{2n+1}^M| \cdot |B_{2m+1}^M| \cos[2(n-m)(\phi+\zeta)] \right] \qquad (4.6)$$

which is positive definite at any point of the phase space. We can obtain the expectation value of the density matrix $\hat{\rho}$ with respect to the coherent state α by using equation (4.7), where $\langle\alpha|\hat{\rho}|\alpha\rangle = \pi Q(\alpha)$.

Finally, for $s = 0$, we have

$$P(\alpha) = \frac{|\lambda|^2}{\pi}\Bigg[\sum_{n=0}^{[(M-1)/2]} |B^M_{2n+1}|^2 L_{2n+1}\left(\frac{\partial^2}{\partial\alpha\partial(-\alpha^*)}\right)\delta(\alpha)$$
$$+ 2\sum_{\substack{m,n=0\\ m>n}}^{[(M-1)/2]} \sqrt{\frac{(2m+1)!}{(2n+1)!}}\,|B^M_{2n+1}|\cdot|B^M_{2m+1}|$$
$$\times \frac{\partial^{2(n-m)}}{\partial(\alpha^{n-m})\partial(-\alpha^*)^{(n-m)}}\, L^{2(m-n)}_{2n+1}$$
$$\times\left(\frac{\partial^2}{\partial\alpha\partial(-\alpha^*)}\right)\delta(\alpha)\cos[2(n-m)(\phi+\zeta)]\Bigg] \tag{4.7}$$

Now we calculate the probability distribution function $\tilde{P}(x)$ by integrating $W(\alpha)$ with $\alpha = x + iy$ over the imaginary variable y, where[(25)]

$$\tilde{P}(x) = \int_{-\infty}^{\infty} W(x+iy)\, dy \tag{4.8}$$

Substituting (4.6) into (4.9), we get

$$\tilde{P}(x) = \sqrt{\frac{2}{\pi}}\,|\lambda|^2\exp(-2x^2)\Bigg[\sum_{n=0}^{[(M-2)/2]} |B^M_{2n+1}|^2\,\frac{2^{-2n-1}H^2_{2n+1}(\sqrt{2}x)}{(2n+1)!}$$
$$+ 2\sum_{\substack{m,n=0\\ m>n}}^{[(M-1)/2]} |B^M_{2m+1}B^M_{2n+1}|\cos[2(n-m)(\phi)]$$
$$\times \frac{2^{-(m+n+1)}H_{2m+1}(\sqrt{2}x)H_{2n+1}(\sqrt{2}x)}{\sqrt{(2n+1)!\,(2m+1)!}}\Bigg] \tag{4.9}$$

where $H_m(z)$ is the Hermite polynomial of order m:

$$H_m(z) = \sum_{r=0}^{[m/2]} \frac{(-1)^r m!\,(2z)^{m-2r}}{r!\,((m-2r)!} \tag{4.10}$$

Figures 4a and 4b plot the Wigner function $W(\alpha)$ for different values of η and M. We find that, when η is small ($\eta = 0.1$) and $M = 5$, the function $W(\alpha)$ has a hole on the summit similar to that of the geometric state; see refs. 7 and 8. When we increase the value of η such that $\eta = 0.6$ keeping the value of M small ($M = 5$), then we have four asymmetric peaks with a chaotic behavior, where we can see the interference between the component states results in the selective preservation of nonclassical effects during the amplification process; see Fig. 4a. On the other hand, if M increases ($M = 17$) with the same value of η ($\eta = 0.6$), the four peaks are shifted and the

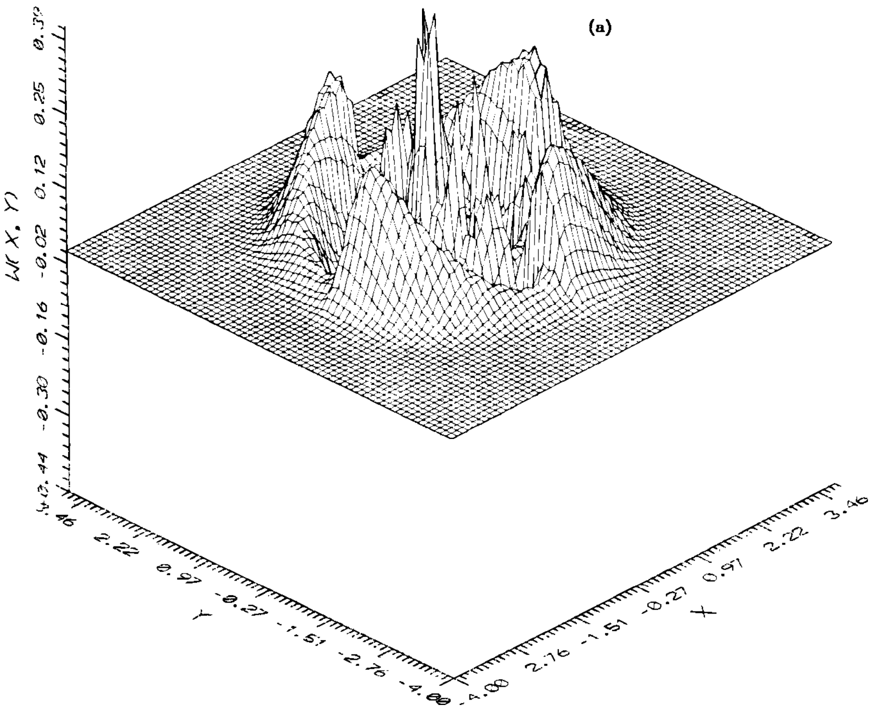


Fig. 4. W-function from equation (4.6) for (a) $M = 5$, $\eta = 0.6$ and (b) $M = 17$, $\eta = 0.6$.

chaotic behavior becomes pronounced (Fig. 4b). For the $Q(\alpha)$ function, we find that when η is small ($\eta = 0.1$) and $M = 5$, the Q-function represents the case of the Fock state $|5\rangle$; see ref. 2. However, if we increase the value of η such that $\eta = 0.6$, then the probability of having single photons also increases, where we have four adjacent deformed peaks at the center (Fig. 5a). As M and η further increase ($M = 17$, $\eta = 0.9$) the four peaks shift (Fig. 5b).

5. APPLICATIONS TO DYNAMICAL SYSTEM

5.1. Jaynes–Cummings Model

In this section we examine the effects of the odd binomial state by two examples of quantum optical systems, the Jaynes–Cummings model (JCM) and the resonance fluorescence. In the present subsection we examine the dynamical evolution of a system of a field starting from an odd binomial state and a two-level atom. The model Hamiltonian representing such a system is known as a Jaynes–Cummings model (model of interaction between

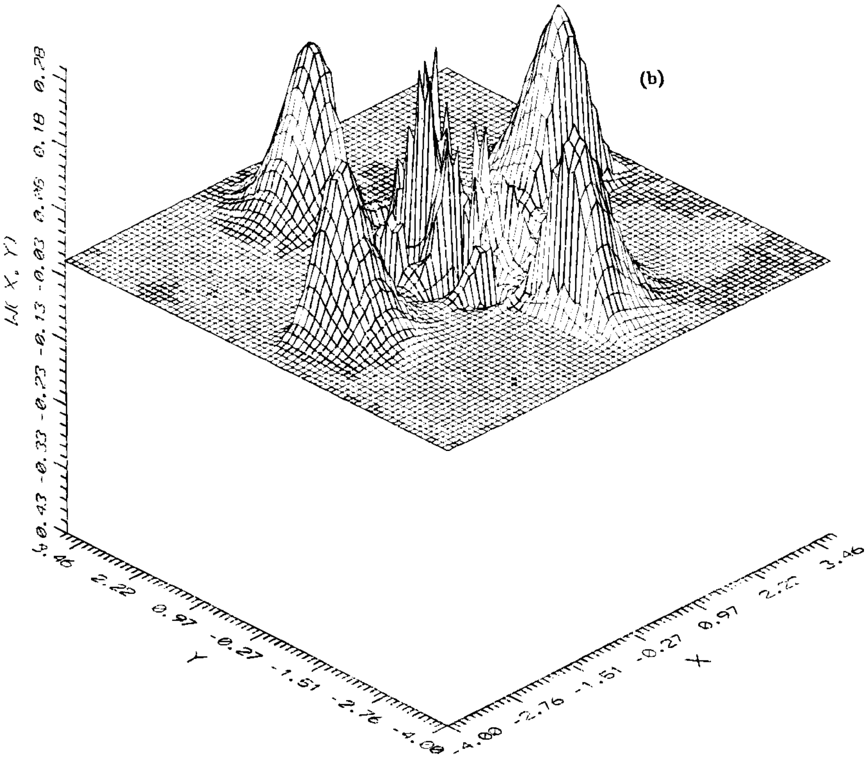


Fig. 4. Continued.

an electromagnetic field and a two-level atom). This model can be written in the RWA at exact resonance as follows:

$$\frac{H}{\hbar} = \omega(a^{\dagger}a + S_z) + g(S_+a + S_-a^{\dagger}) \tag{5.1}$$

The atom is described by the Pauli operators $S_{\pm}$, S_z; g is the coupling constant and ω is the atomic transition frequency, which has been taken equal to the frequency of the radiation mode. The atomic operators satisfy the relations

$$[S_z, S_{\pm}] = \pm S_{\pm} \tag{5.2}$$

$$[S_+, S_-] = 2S_z, \qquad S_+^2 = S_-^2 = 0$$

If we define $\hat{C}_1$ and $\hat{C}_2$, such that

$$\hat{C}_1 = a^{\dagger}a + S_z \tag{5.3a}$$

$$\hat{C}_2 = g(S_+a + S_-a^{\dagger}) \tag{5.3b}$$

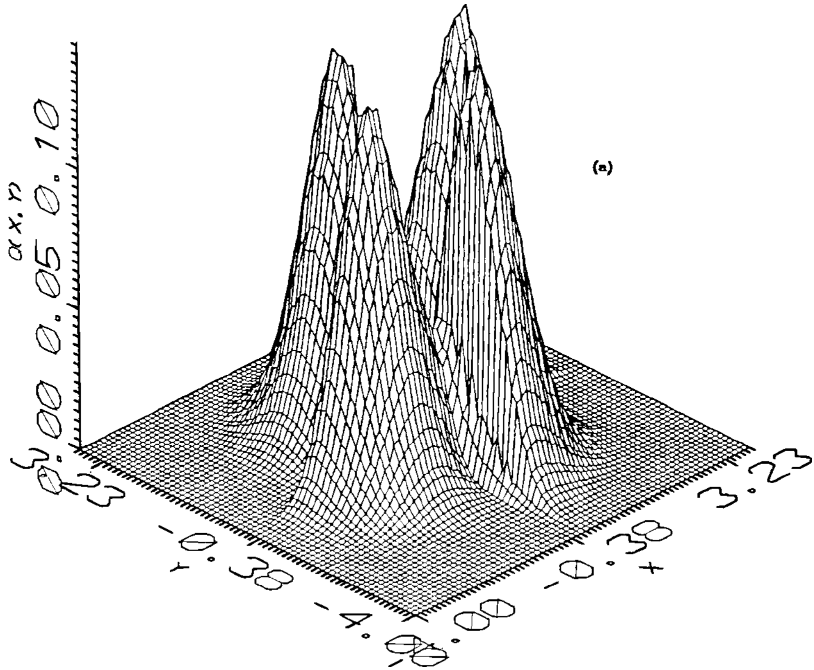


Fig. 5. Q-function from equation (4.7) for (a) $M = 5$, $\eta = 0.6$, (b) $M = 17$, $\eta = 0.9$.

then it is easy to show that $\hat{C}_1$ and $\hat{C}_2$ are constants of motion ($d\hat{C}_i/dt = 0$, $i = 1, 2$). The evolution operator for the interaction part $\hat{U}_i(t, 0)$ is given by

$$\hat{U}_i(t, 0) = \cos \hat{D}t - i \frac{\sin \hat{D}t}{\hat{D}} \hat{C}_2 \tag{5.4}$$

with $\hat{D} = \sqrt{\hat{C}_2^2}$. Now let us suppose that the radiation field is prepared to be initially in the odd binomial state, and the atom in the atomic superposition coherent state,[27] such that

$$|\theta, \nu\rangle = \cos \theta|+1\rangle + e^{-i\nu} \sin \theta|-1\rangle \tag{5.5}$$

where the states $|+1\rangle$ and $|-1\rangle$ are the atomic excited and ground states, respectively, while ν, and θ are two different phases. From equations (1.1) and (5.4) together with equation (5.5) one can write the time-dependent eigenstate of the system as

$$|\psi_i(t)\rangle = \lambda \sum_{n=0}^{[(M-1)/2]} B_{2n+1}^{M}[h_1(n, t)| + 1, 2n + 1\rangle + h_2(n, t)| + 1, 2n\rangle$$
$$+ h_3(n, t)| - 1, 2n + 1\rangle + h_4(n, t)| - 1, 2n + 2\rangle] \tag{5.6}$$

The coefficients $h_\tau(n, t)$ with $r = 1, 2, 3$, and 4 are given by

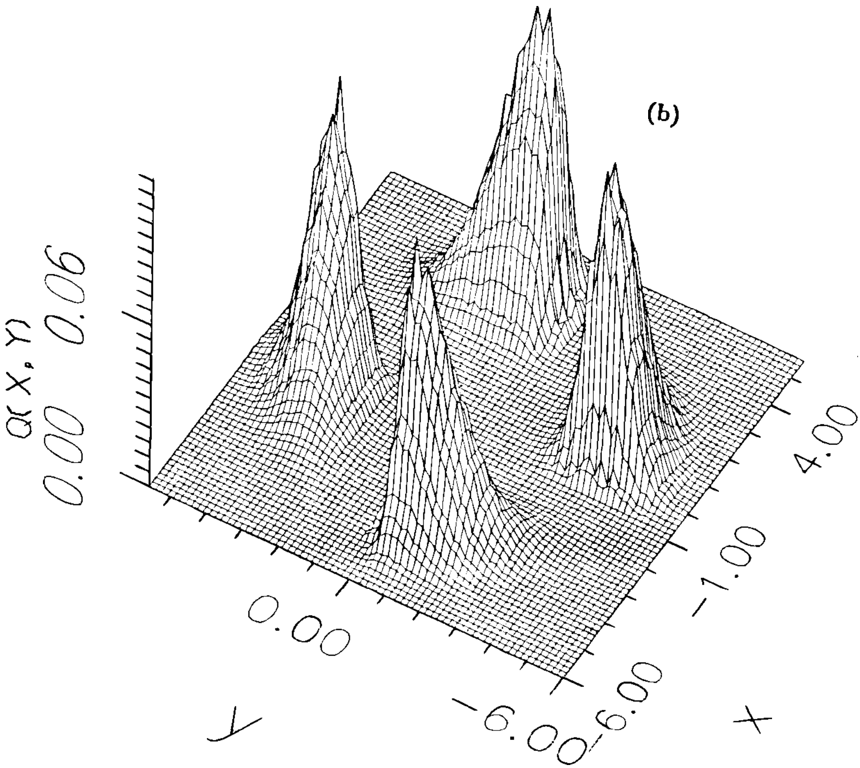


Fig. 5. Continued.

$$h_1 = \cos\theta \cos(T\sqrt{2n+2}) \tag{5.7a}$$

$$h_2 = -ie^{-i\nu} \sin\theta \sin(T\sqrt{2n+1}) \tag{5.7b}$$

$$h_3 = e^{-i\nu} \sin\theta \cos(T\sqrt{2n+1}) \tag{5.7c}$$

$$h_4 = -i\cos\theta \sin(T\sqrt{2n+2}) \tag{5.7d}$$

where $T = gt$. From the above equations we can calculate the expectation values for different operators. Thus

$$\langle a^{+}\rangle = \langle a\rangle^{*} = 0 \tag{5.8}$$

$$\begin{aligned}\langle a^{\dagger} a\rangle = {} & |\lambda|^2|\eta|^2(M/2)[1 + (1 - 2|\eta|^2)^{(M-1)}] \\ & + |\lambda|^2 \sum_{n=0}^{[(M-1)/2]} |B_{2n+1}^{M}|^2[\cos^2\theta \sin^2(T\sqrt{2n+2}) \\ & - \sin^2\theta \sin^2(T\sqrt{2n+1})]\end{aligned} \tag{5.9}$$

$$\langle a^{\dagger 2}a^2\rangle = |\lambda|^2|\eta|^4(M/2)(M-2)[1-(1-2|\eta|^2)^{(M-2)}]$$
$$+ 2|\lambda|^2 \sum_{n=0}^{[(M-1)/2]} |B_{2n+1}^M|^2 \Bigg[(2n+1)\cos^2\theta \sin^2(T\sqrt{2n+2})$$
$$- 2n\sin^2\theta \sin^2(T\sqrt{2n+1}) \Bigg] \tag{5.10}$$

$$\langle a^{2s}\rangle = \frac{|\lambda|^2\eta^{2s}}{(1-|\eta|^2)^s} \sum_{n=0}^{[(M-1)/2]} |B_{2n+1}^M|^2 \sqrt{\frac{(M-2n-1)!}{(M-2n-2s-1)!}} \Bigg\{ \cos^2\theta$$
$$\times \Bigg[\cos(T\sqrt{2n+2+2s})\cos(T\sqrt{2n+2})$$
$$+ \sqrt{\frac{2n+2s+2}{2n+2}} \sin(T\sqrt{2n+2s+2})\sin(T\sqrt{2n+2}) \Bigg]$$
$$+ \sin^2\theta \Bigg[\sqrt{\frac{2n+1}{2n+2s+1}} \sin(T\sqrt{2n+1})\sin(T\sqrt{2n+2s+1})$$
$$+ \cos(T\sqrt{2n+2s+1})\cos(T\sqrt{2n+1}) \Bigg] \Bigg\} \tag{5.11}$$

Having obtained the expectation values of the operators, we are in a position to discuss the time-dependent normalized second-order correlation function as well as time-dependent squeezing phenomena. This is done in the next subsections.

5.1.1. Time-Dependent Second-Order Correlation Function

Now if we use equations (2.4) and (5.9) together with equation (5.10) we can calculate the correlation function $g^{(2)}(T)$. Figures 6a and 6b plot this function against the scaled time T in two different cases, the atom initially in the excited state where $\theta = 0$, or the atom initially in the ground state, where $\theta = \pi/2$. By taking the mean photon number $\bar{n} = 5$ in the above cases we see that the behavior of the function $g^{(2)}(T)$ in both cases is in general oscillatory; when we take the value of M to be small ($M = 9$) in both cases, the correlation function shows sub-Poissonian behavior, while if we increase the value of M ($M = 19$), we find that the correlation function for the case when the atom is in the excited state still shows subPoissonian behavior, while the function for the ground-state case starts to oscillate, showing super- and sub-Poissonian as well as partially coherence behavior. By increasing the value of M up to 29, we notice that the correlation function in the excited

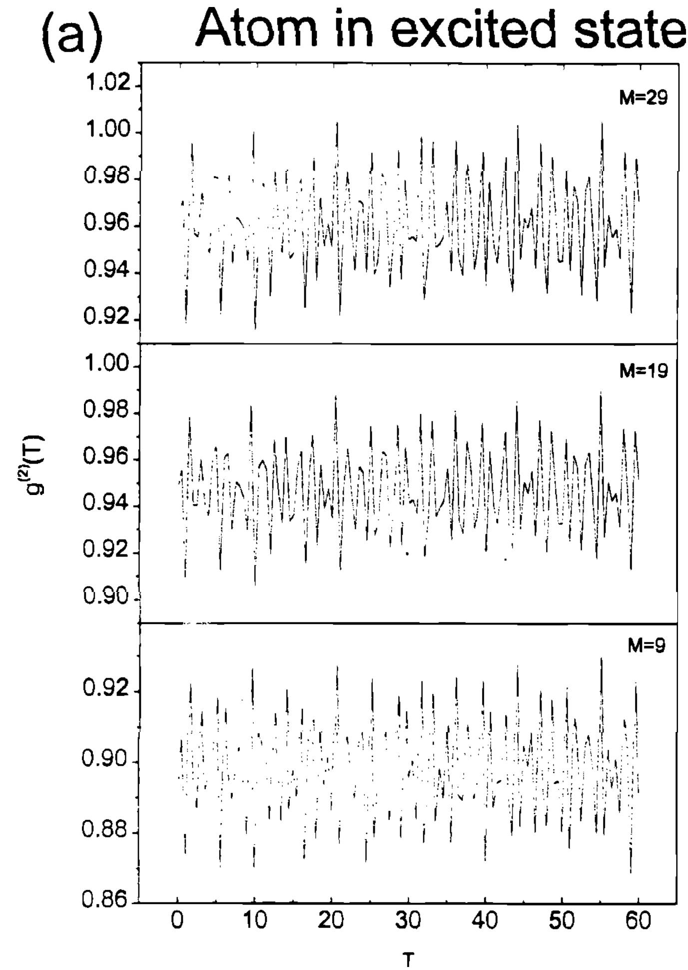


Fig. 6. Time-dependent normalized second-order correlation function $g^{(2)}(T)$ for (a) $\theta = 0$ (atom initially in excited state) and (b) $\theta = \pi/2$ (atom initially in the ground state).

state starts to show super-Poissonian behavior similar to that of the ground-state case; however, in the ground-state case the super-Poissonian behavior is more pronounced than in the excited-state case (Figs. 6a and 6b).

5.1.2. Time-Dependent Squeezing Phenomena

In order to discuss time-dependent squeezing (normal squeezing) we have to examine the quadrature variances $S_1(T)$ and $S_2(T)$ against the normal-

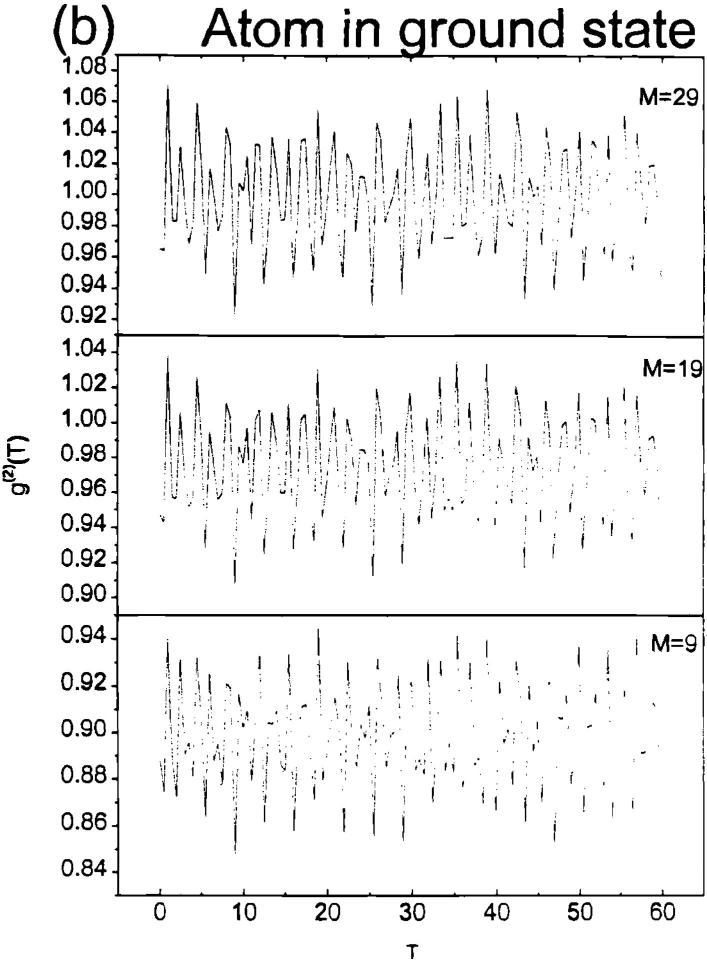


Fig. 6. Continued.

ized time T. By using equations (2.8), (5.9), and (5.11) we plot the quadrature variances $S_1(T)$ and $S_2(T)$ against the normalized time for $M = 3$ and $\eta = 0.6$ in Fig. 7. In this figure we have considered the atom to be perpared initially in the ground state where $\theta = \pi/2$. It is easy to see that, although the first quadrature $S_1(T)$ includes a periodic function, the squeezing is occurs once for a short period of time, while the squeezing in the second quadrature $S_2(T)$ appears frequently. On the other hand, when $\theta = 0$ (atom initially in the excited state) we observe no squeezing in both quadratures.

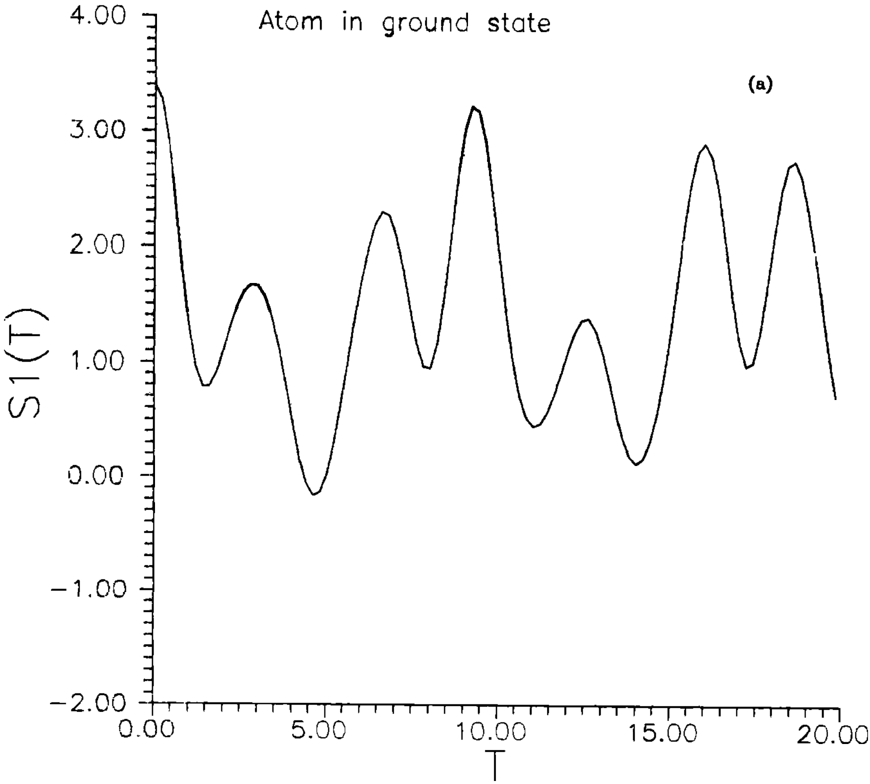


Fig. 7. Time-dependent normal squeezing when $\theta = \pi/2$ (atom initially in ground state) for (a) $S_1(T)$ and (b) $S_2(T)$.

5.2. Resonance Fluorescence

The resonance fluorescence phenomenon is related to a radiatively decaying two-level atomic system coupled to an external radiation field in free space. In the following we are concerned with the steady-state regime ($t \to \infty$) in two different cases: the single atom ($N = 1$), and the thermodynamic limit in a cooperative many-atom system where $N \to \infty$.

5.2.1. A Single Atom

It is well known that when we have an external field in the Fock state $|n\rangle$ then in the steady-state case the mean atomic inversion for a single two-level atom interacting with this field takes the form[28,29]

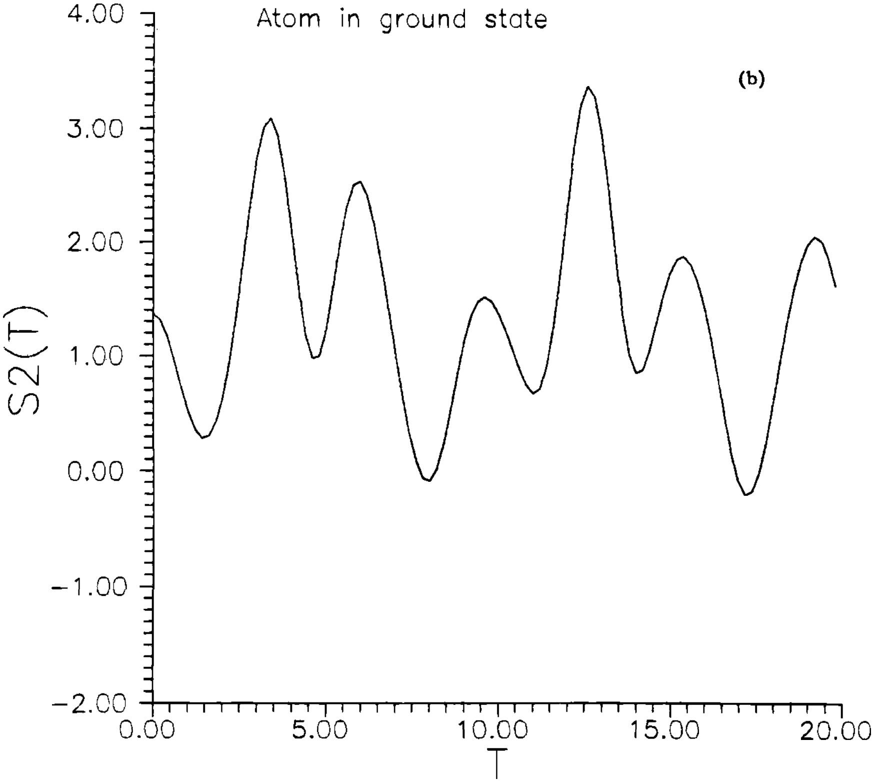


Fig. 7. Continued.

$$\langle S_z(\infty)\rangle_n = -\frac{1}{2}\sum_{m=0}^{n}\frac{(-1)^m n!\,(b^2)^m}{(n-m)!} = -\left(\frac{1}{2}\right)(b^2)^n L_n^{(-n-1)}(-b^{-2}) \quad (5.12)$$

where

$$b^2 = 2\hbar^{-2}g^2\left(\Delta^2 + \frac{1}{4}\gamma^2\right)^{-1} \quad (5.13)$$

g is the coupling constant, and γ is the Einstein coefficient, while Δ is the frequency detuning between the external field and the atom.

Now when we consider the statistical average over the state of the field which is taken to be the odd-binomial state (1.1), then in the steady-state case ($t \to \infty$), the mean atomic inversion is given by

$$\langle S_z\rangle = |\lambda|^2 \sum_{n-0}^{\left[\frac{M-1}{2}\right]} |B_{2n+1}^M|^2 \times [\langle S_z(\infty)\rangle_{2n+1} \cos^2\theta - \langle S_z(\infty)\rangle_{2n} \sin^2\theta] \tag{5.14}$$

where $\langle S_z(\infty)\rangle_{2n}$ and $\langle S_z(\infty)\rangle_{2n+1}$ are given by (5.12). For the odd coherent state equation (5.14) reduces to

$$\begin{aligned}\langle S_z(\infty)\rangle = -\frac{1}{2\sinh|\alpha|^3}\Bigg[&\frac{(\sinh|\alpha|^2 - b^2|\alpha|^2\cosh|\alpha|^2)\cos^2\theta}{1 - b^4|\alpha|^4} \\ &- (\sin^2\theta)[(\sinh|\alpha|^2)(1 - D'^2)^{-1}[1 + b^2D'|\alpha|^2] \\ &- b^2(\cosh|\alpha|^2)(1 - D'^2)^{-1}|\alpha|^2(b^2|\alpha|^2 + D')] \\ &\times (1 - b^4|\alpha|^4)^{-1}\Bigg]\end{aligned} \tag{5.15}$$

where $D' = d/d|\alpha|^2$.

5.2.2. Thermodynamic Limit

Now consider the case of resonance fluorescence in which $N \to \infty$ and $\gamma \to 0$ such that (γN) is finite; then at exact resonance we have the scaled atomic inversion in the number state of the field in the form[20,21]

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{\langle S_z(\infty)\rangle_n}{N}\right] &= -\frac{1}{2}C_n\left(\frac{1}{2}; X^2\right) \\ &= -\frac{1}{2}\sum_{m=0}^{n}\binom{1/2}{m}\binom{n}{m} m!\,(-X^{-2})^m\end{aligned} \tag{5.16}$$

where $X^2 = \gamma n/(2\hbar^{-2}\gamma^2)$ and C_n are the Poisson–Charlier polynomials. For the state (5.6) the thermodynamic limit becomes

$$\begin{aligned}\lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle}{N}\right] = \frac{|\lambda|^2}{2}&\sum_{n=0}^{\left[\frac{M-1}{2}\right]} |B_{2n+1}^M|^2 \\ &\times\left[C_{2n}\left(\frac{1}{2}; X^2\right)\sin^2\theta - C_{2n+1}\left(\frac{1}{2}, X^2\right)\cos^2\theta\right]\end{aligned} \tag{5.17}$$

In Fig. 8 we plot the atomic inversion for a single atom given by equation (5.14) (full line) and the thermodynamic limit given by equation (5.17) (dashed line) against the parameter η for $\theta = 0$ and $\pi/2$ with $b^2 = X^{-2} = 10^{-2}$.

We observe that when the atom is in the excited state (Fig. 8a) the value of

$$\lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle}{N}\right] \cong 0$$

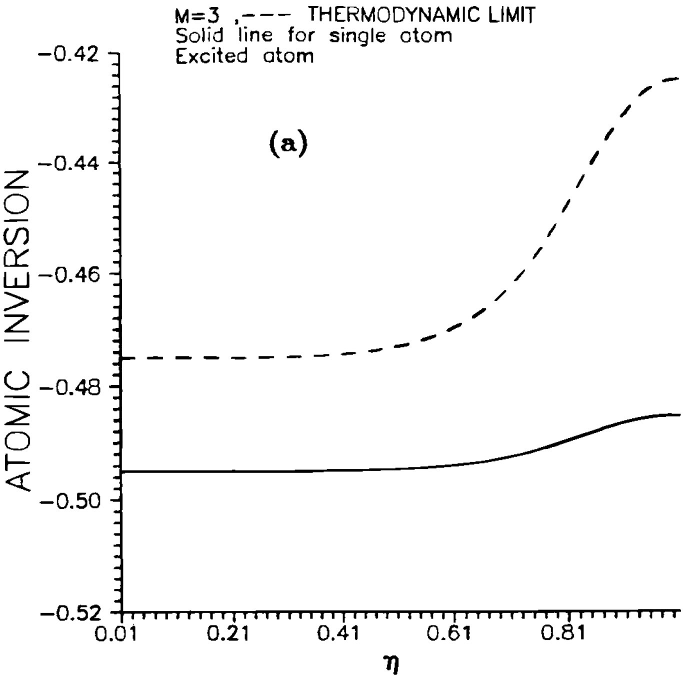
is achieved very slowly and it needs larger values of M and η. This behavior also has been noticed in the case of the generalized geometric state.[4] For an atom in the ground state (Fig. 8b), upon increasing the values of η and M, $\langle S_z(\infty)\rangle$ reaches its steady-state value at a slower rate. The same behavior is also observed in ref. 4. It is clear that the rate in the thermodynamic limit is slower than that of a single atom when the atom is in the ground state (Fig. 8b) and conversely when the atom is in the excited state (Fig. 8a). For the odd coherent state, we thus have

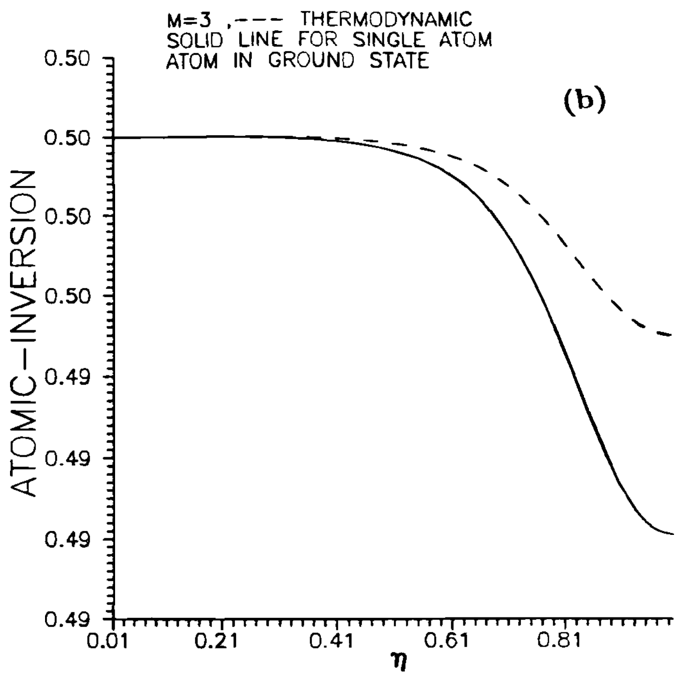
$$\lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle}{N}\right] \to \frac{1}{2\sinh|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{4n+2}}{(2n+1)!} \times\left[C_{2n}\left(\frac{1}{2};X^2\right)\sin^2\theta - C_{2n+1}\left(\frac{1}{2},X^2\right)\cos^2\theta\right] \tag{5.18}$$

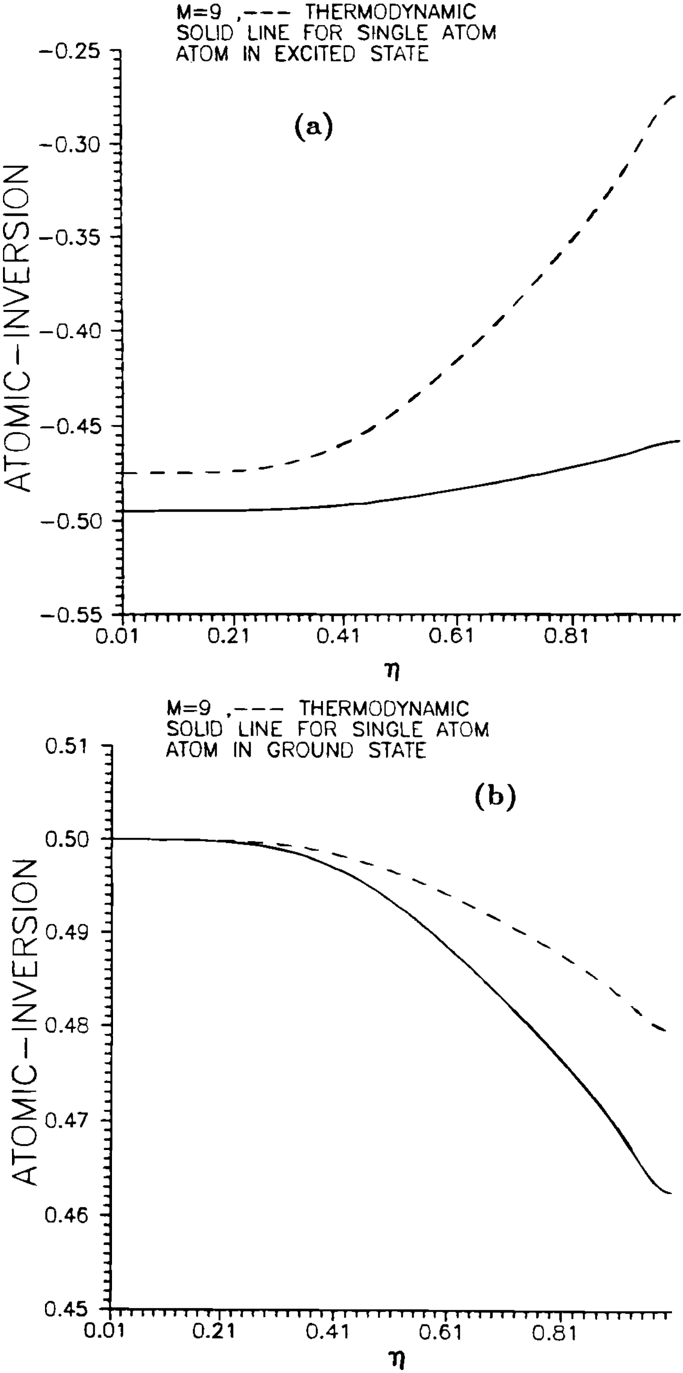
6. CONCLUSION

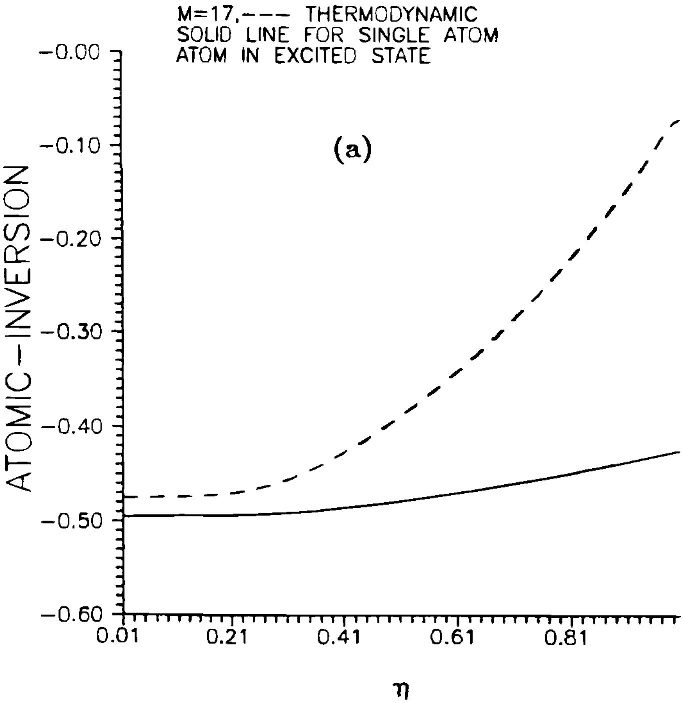
In the present paper we have introduced the odd binomial state and considered some of its statistical properties. In addition to the oscillation behavior in the photon number distribution which could be expected due to removing the even number states for the photons, we have discussed the Glauber second-order correlation function. States of this type show sub-Poissonian behavior for all values of η and M. However, for large values of M the states approach the value 1 as η becomes closer to unity, giving a higher degree of coherence. On the other hand, normal squeezing has been found to be absent for the odd binomial state whatever the values of η and M. Amplitude-squared squeezing exists for the same state, and becomes more pronounced for larger values of M with the point of maximum squeezing moving toward higher values of η. Investigation of the phase distribution function in the Pegg–Barnett formalism shows a loss of phase information as $\eta \to 0$ or $\eta \to 1$ which reflects the fact of having a Fock state. But for

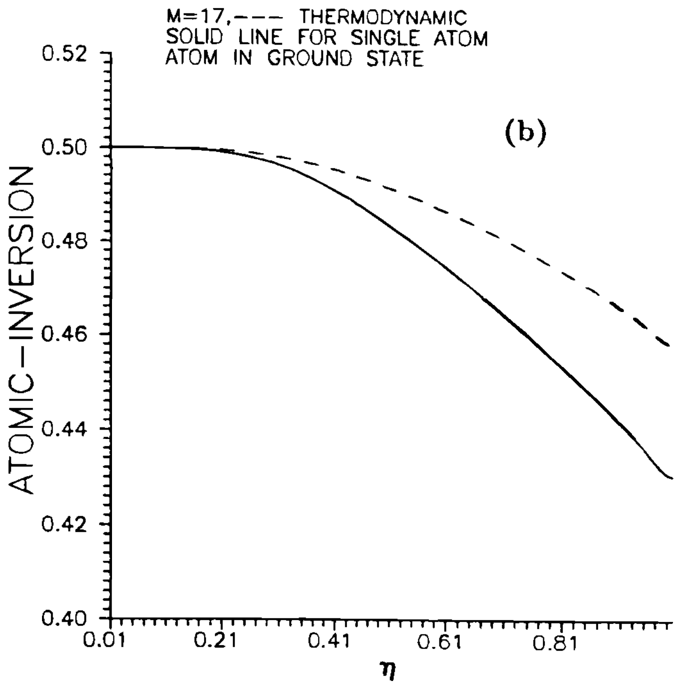
Fig. 8. Atomic inversion in the steady state for resonance fluorescence given by equations (5.15) (single atom, full line) and (5.17) (thermodynamic limit, dashed line) for (a) $\theta = 0$ (atom initially in the excited state) and (b) $\theta = \pi/2$ (atom initially in the ground state).


M=3 ,--- THERMODYNAMIC LIMIT
Solid line for single atom
Excited atom
(a)
ATOMIC INVERSION
−0.42
−0.44
−0.46
−0.48
−0.50
−0.52
0.01
0.21
0.41
0.61
0.81
η


M=3 ,--- THERMODYNAMIC
SOLID LINE FOR SINGLE ATOM
ATOM IN GROUND STATE
(b)
ATOMIC−INVERSION
0.50
0.50
0.50
0.50
0.49
0.49
0.49
0.49
0.01
0.21
0.41
0.61
0.81
η


M=9 ,--- THERMODYNAMIC
SOLID LINE FOR SINGLE ATOM
ATOM IN EXCITED STATE
(a)
ATOMIC-INVERSION
−0.25
−0.30
−0.35
−0.40
−0.45
−0.50
−0.55
0.01
0.21
0.41
0.61
0.81
η
M=9 ,--- THERMODYNAMIC
SOLID LINE FOR SINGLE ATOM
ATOM IN GROUND STATE
(b)
ATOMIC-INVERSION
0.51
0.50
0.49
0.48
0.47
0.46
0.45
0.01
0.21
0.41
0.61
0.81
η


M=17,--- THERMODYNAMIC
SOLID LINE FOR SINGLE ATOM
ATOM IN EXCITED STATE
(a)
ATOMIC-INVERSION
-0.00
-0.10
-0.20
-0.30
-0.40
-0.50
-0.60
0.01
0.21
0.41
0.61
0.81
η


M=17,--- THERMODYNAMIC
SOLID LINE FOR SINGLE ATOM
ATOM IN GROUND STATE
(b)
ATOMIC-INVERSION
0.52
0.50
0.48
0.46
0.44
0.42
0.40
0.01
0.21
0.41
0.61
0.81
η

$0 < \eta < 1$ peak occurs around $\theta = 0$ and two wings at $\theta = \pm\pi$, as shown in earlier investigations.[9,14] Nonclassical effects can be studied through the quasiprobability functions. A negative value for the Wigner function is a signature for nonclassical effects of the state. This has been demonstrated for the Wigner function for the odd-binomial state, also an interference pattern is apparant due to the superposition of the two binomial states. These effects are also demonstrated for the Q-function. Increasing the number M adds more excitations and the interference pattern becomes richer. The dynamical behavior of a two-level atom in a cavity with a field prepared initially in an odd binomial state shows some interesting features. In contrast to the initial sub-Poissonian behavior of the field, the existence of the atom changes its characteristics; it becomes oscillatory and shows super-Poissonian behavior especially when the atom starts from a ground state. The phenomenon of normal squeezing which is absent for the field initially is found to exist as time develops, especially when the atom starts from its ground state. However, the state of the field stays unsqueezed when the atom is prepared initially in its excited state. Numerical investigations for amplitude-squared squeezing show that it exists in small amounts in both components for small values of η and large values of M. Therefore the existence of the two-level atom in a field prepared initially in an odd binomial state dramatically changes its charactristics. Finally, the atomic inversion in the steady state for resonance fluorescence for a single atom and the thermodynamic limit approaches the saturated case as M increases. The rate in the thermodynamic limit is slower than for the case of a single atom when the atom is in its ground state and conversely when the atom is in its excited state.

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